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ON THE HIGH ORDER EFFECTS IN THE METHODS OF KRYLOV-BOGOLIUBOV AND POINCARÉ

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ABSTRACT

In this article the formal process for the determination of the higher order perturbations of the elements is developed using the methods of Krylov-Bogoliubov and Poincaré. Such a development is necessary, for example, in the lunar problem where very high order perturbations have to be determined. The differential equations are formed for the elements which are affected only by the long period and the secular terms. The determination of these elements, as well as the elimination of the short period effects, is reduced to the solving of a set of partial differential equations, step by step. By developing the displacement operator into a series of the differential operators of Faa de Bruno we can write these equations in a concise form.

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ON THE HIGH ORDER EFFECTS IN THE METHODS OF KRYLOV-BOGOLIUBOV AND POINCARÉ

INTRODUCTION

In this article we develop the formalism for the determination of the general perturbations of higher orders in Celestial Mechanics by means of the method of Krylov-Bogoliubov (1961) and of the method of Poincaré (1892) and von Zeipel (1916). The solution of the problem is obtained in terms of the Krylov-Bogoliubov averaging operator, of Faa de Bruno differential operators (1855) and the integrating operator.

In the method of Krylov-Bogoliubov, as in the method of Poincaré, the final goal is the elimination of the short period effects and the derivation of the elements affected only by the long period and the secular perturbations. The original work of Krylov and Bogoliubov was influenced by the problems of Celestial Mechanics. If one looks closely at the method of LeVerrier (1856) of the general perturbations, he can easily recognize the same basic idea, but the method of Krylov and Bogoliubov achieved its fame under its present name because of its extensive application to the problems of Theoretical Physics. In the majority of these problems, there is no need for the computation of the effects of higher orders. Normally, only the effects of the first and of the second orders, rarely of the third order, are computed. The standard presentation of the method does not go beyond these limits. This accuracy is insufficient from the standpoint of Celestial Mechanics. In the lunar problem, we must go up to the perturbations of the ninth order with respect to the ratio of mean motions of the satellite and of the sun, if we want to secure the necessary accuracy of the long period terms.

Thus, the formalism of the Krylov-Bogoliubov method must be extended in order to cover such cases and especially to provide for the determination of the long period effects of higher orders. The long period and the secular effects are chiefly responsible for the behavior and stability of orbits of the celestial body and their accurate determination is of great importance. The positive characteristic of the method of Krylov and Bogoliubov is that the canonical form of equations of motion is not required and thus it is applicable to a much wider range of problems than is the method of Poincaré.

However, the number of the partial differential equations to be solved in the process of elimination of the short period terms increases, as compared to the method of Poincaré. It is the price we pay for the extension of the domain of applicability.

In the method of Poincaré the equations of motion have the canonical form and the problem of the elimination of the short period terms from the coordinates and the momenta is reduced to the elimination of such terms from the Hamiltonian by means of a properly chosen canonical transformation. Assuming that the characteristic function S of this transformation is developable into a power series with respect to a small parameter, we reduce the determination of S to the solution of a chain of partial differential equations, step by step. Recently, Giacaglia (1964) has established the general form of these equations.

We show in this work, that the partial differential equations of the method by Poincaré take a specially concise form if written in terms of Faa de Bruno operators (1855).

THE HIGHER ORDER PERTURBATIONS IN KYRLOV-BOGOLIUBOV METHOD

Let us consider the system of the vectorial differential equations

$$\frac{dx}{dt} = X(x;\ y,\ \eta), \tag{1}$$

$$\frac{dy}{dt} = \lambda(x) + Y(x;\ y,\ \eta), \tag{2}$$

$$\frac{d\eta}{dt} = H(x;\ y,\ \eta), \tag{3}$$

where X, Y, H are periodic in y and η with the period 2π in each component. These vectors are assumed to be developable in powers of a small parameter. We have

$$x = \sum_{j=1}^{\infty} X_j\,(x;\ y,\ \eta), \tag{4}$$

$$Y = \sum_{j=1}^{\infty} Y_j\,(x;\ y,\ \eta), \tag{5}$$

$$H = \sum_{j=1}^{\infty} H_j\,(x;\ y,\ \eta), \tag{6}$$

where the functions X_j, Y_j, H_j are of the form

$$F = \sum_{n,\,v} F_{n,\,v}\,(x)\,\exp\, i\,(n\cdot y + v\cdot\eta), \tag{7}$$

where n and v are vectors whose components are integers.

The terms in (7) are:

the short periodic if $n \neq 0$,

the long periodic if $n = 0$, but $v \neq 0$

the secular if $n = 0$ and $v = 0$.

The averaging operator M performs the extraction of the long period and the secular terms from (7). Thus

$$MF = \sum_{\nu} F_{0,\nu}(x) \exp i\, \nu \cdot \eta$$

In addition to the Krylov-Bogoliubov operator M it is also convenient to make use of the operator P which performs the extraction of the short period terms only,

$$PF = \sum_{n \neq 0, \nu} F_{n,\nu}(x) \exp i\,(n \cdot y + \nu \cdot \eta)$$

In the further exposition we make use of the partial del-operators

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \eta}$$

and introduce the partial differential equation of the form

$$\lambda(x) \cdot \frac{\partial \psi}{\partial y} = PF$$

Evidently

$$\psi = \sum_{n \neq 0, \nu} \frac{F_{n,\nu}}{\lambda, n} \exp \cdot i\,(n \cdot y + \nu \cdot \eta)$$

Introducing the integrating operator Q we can write

$$\psi = QPF$$

Let us determine the transformation

$$x = x^* + a(x^*;\, y^*,\, \eta^*) \tag{8}$$

$$y = y^* + b(x^*;\, y^*,\, \eta^*) \tag{9}$$

$$\eta = \eta^* + \beta(x^*;\, y^*,\, \eta^*) \tag{10}$$

in such a way that the differential equations for the new variables

$$\frac{dx^*}{dt} = X^* (x^*; \ -, \ \eta^*) \tag{11}$$

$$\frac{dy^*}{dt} = \lambda(x^*) + Y^*(x^*; \ -, \ \eta^*) \tag{12}$$

$$\frac{d\eta^*}{dt} = H^* (x^*; \ -, \ \eta^*) \tag{13}$$

do not contain the new short period argument y^*. We put a dash in place of y^* in order to emphasize its absence.

We shall determine the formal developments

$$a = \sum_{j=1}^{\infty} a_j, \quad b = \sum_{j=1}^{\infty} b_j, \quad \beta = \sum_{j=1}^{\infty} \beta_j, \tag{14}$$

$$X^* = \sum_{j=1}^{\infty} X_j^*, \quad Y^* = \sum_{j=1}^{\infty} Y_j^*, \quad H^* = \sum_{j=1}^{\infty} H_j^* \tag{15}$$

in such a way that the equations (11) - (13) have the prescribed form. It follows from these equations that the operator d/dt can be written in the form

$$\frac{d}{dt} = \lambda(x^*) \cdot \frac{\partial}{\partial y^*} + D,$$

where

$$D = X^* \cdot \frac{\partial}{\partial x^*} + Y^* \cdot \frac{\partial}{\partial y^*} + H^* \cdot \frac{\partial}{\partial \eta^*}$$

From (8) - (10) and (11) - (13) we have:

$$\frac{dx}{dt} = X^* + \lambda \cdot \frac{\partial a}{\partial y^*} + D\,a \tag{16}$$

$$\frac{dy}{dt} = \lambda + Y^* + \lambda \cdot \frac{\partial b}{\partial y^*} + D\,b, \tag{17}$$

$$\frac{d\eta}{dt} = H^* + \lambda \cdot \frac{\partial \beta}{\partial y^*} + D\beta. \tag{18}$$

Introducing the displacement operator

$$1 + T(x^*;\ y^*,\ \eta^*) = \exp\left(a \cdot \frac{\partial}{\partial x^*} + b \cdot \frac{\partial}{\partial y^*} + \beta \cdot \frac{\partial}{\partial \eta^*}\right)$$

We can write (1) - (3) as

$$\frac{dx}{dt} = (1 + T)\, X(x^*;\ y^*,\ \eta^*), \tag{19}$$

$$\frac{dy}{dt} = (1 + T)\ [\lambda(x^*) + Y(x^*;\ y^*,\ \eta^*)], \tag{20}$$

$$\frac{d\eta}{dt} = (1 + T)\, H(x^*;\ y^*,\ \eta^*). \tag{21}$$

Comparing (16) - (18) with (19) - (21) and changing the notation we have

$$\lambda \cdot \frac{\partial a}{\partial y} = (X - X^*) + (TX - Da), \tag{22}$$

$$\lambda \cdot \frac{\partial b}{\partial y} = (Y - Y^*) + T\lambda + (TY - Db), \tag{23}$$

$$\lambda \cdot \frac{\partial \beta}{\partial y} = (H - H^*) + (TH - D\beta). \tag{24}$$

Making use of (14) we can represent $1 + T$ in the form

$$1 + T = \exp . \sum_{j=1}^{\infty} \delta_j = \sum_{j=0}^{\infty} T_j, \tag{25}$$

where we put

$$\delta_j = a_j \cdot \frac{\partial}{\partial x} + b_j \cdot \frac{\partial}{\partial y} + \beta_j \cdot \frac{\partial}{\partial \eta}.$$

The operators T_j are polynomials in $\delta_1,\ \delta_2, \ldots$ They can be decomposed into the sums

$$T_j = \sum_{k=1}^{j} T_{j,k},$$

where $T_{j,k}$ are homogeneous of the degree k with respect to the δ-operators. Making use of the expressions obtained by Faa de Bruno (1855) for the higher derivatives of a function depending upon another function, we obtain

$$T_0 = 1$$

$$T_{1,1} = \delta_1$$

$$T_{2,1} = \delta_2$$

$$T_{2,2} = \frac{1}{2}\delta_1^2,$$

$$T_{3,1} = \delta_3$$

$$T_{3,2} = \delta_1 \delta_2$$

$$T_{3,3} = \frac{1}{6}\delta_1^3$$

$$T_{4,1} = \delta_4$$

$$T_{4,2} = \delta_1 \delta_3 + \frac{1}{2}\delta_2^2$$

$$T_{4,3} = \frac{1}{2}\delta_1^2 \delta_2$$

$$T_{4,4} = \frac{1}{24}\delta_1^4$$

$$T_{5,1} = \delta_5$$

$$T_{5,2} = \delta_1 \delta_4 + \delta_2 \delta_3$$

$$T_{5,3} = \frac{1}{2}\delta_1^2 \delta_3 + \frac{1}{2}\delta_1 \delta_2^2$$

$$T_{5,4} = \frac{1}{6}\delta_1^3 \delta_2$$

$$T_{5,5} = \frac{1}{120}\delta_1^5$$

$$T_{6,1} = \delta_6$$

$$T_{6,2} = \delta_1 \delta_5 + \delta_2 \delta_4 + \frac{1}{2}\delta_3^2$$

$$T_{6,3} = \frac{1}{2}\delta_1^2\,\delta_4 + \delta_1\,\delta_2\,\delta_3 + \frac{1}{6}\delta_2^3$$

$$T_{6,4} = \frac{1}{6}\delta_1^3\,\delta_3 + \frac{1}{4}\delta_1^2\,\delta_2^2$$

$$T_{6,5} = \frac{1}{24}\delta_1^4\,\delta_2$$

$$T_{6,6} = \frac{1}{720}\delta_1^6$$

$$T_{7,1} = \delta_7$$

$$T_{7,2} = \delta_1\,\delta_6 + \delta_2\,\delta_5 + \delta_3\,\delta_4$$

$$T_{7,3} = \frac{1}{2}\delta_1^2\,\delta_5 + \delta_1\,\delta_2\,\delta_4 + \frac{1}{2}\delta_1\,\delta_3^2 + \frac{1}{2}\delta_2^2\,\delta_3$$

$$T_{7,4} = \frac{1}{6}\delta_1^3\,\delta_4 + \frac{1}{2}\delta_1^2\,\delta_2\,\delta_3 + \frac{1}{6}\delta_1\,\delta_2^3$$

$$T_{7,5} = \frac{1}{24}\delta_1^4\,\delta_3 + \frac{1}{12}\delta_1^3\,\delta_2^2$$

$$T_{7,6} = \frac{1}{120}\delta_1^5\,\delta_2$$

$$T_{7,7} = \frac{1}{5040}\delta_1^7$$

$$T_{8,1} = \delta_8$$

$$T_{8,2} = \delta_1\,\delta_7 + \delta_2\,\delta_6 + \delta_3\,\delta_5 + \frac{1}{2}\delta_4^2$$

$$T_{8,3} = \frac{1}{2}\delta_1^2\,\delta_6 + \delta_1\,\delta_2\,\delta_5 + \delta_1\,\delta_3\,\delta_4 + \frac{1}{2}\delta_2^2\,\delta_4 + \frac{1}{2}\delta_2\,\delta_3^2$$

$$T_{8,4} = \frac{1}{6}\delta_1^3\,\delta_5 + \frac{1}{2}\delta_1^2\,\delta_2\,\delta_4 + \frac{1}{4}\delta_1^2\,\delta_3^2 + \frac{1}{2}\delta_1\,\delta_2^2\,\delta_3 + \frac{1}{24}\delta_2^4$$

$$T_{8,5} = \frac{1}{24}\delta_1^4\,\delta_4 + \frac{1}{6}\delta_1^3\,\delta_2\,\delta_3 + \frac{1}{12}\delta_1^2\,\delta_2^3$$

$$T_{8,6} = \frac{1}{120}\delta_1^5\,\delta_3 + \frac{1}{48}\delta_1^4\,\delta_2^2$$

$$T_{8,7} = \frac{1}{720}\delta_1^6\,\delta_2$$

$$T_{8,8} = \frac{1}{40320}\delta_1^8.$$

The set of operators T_j given here permits one to develop the general perturbations up to the eighth order. The extension of the given table and the check computations can be performed using the general formulae

$$T_{j,k} = \sum \frac{\delta_1^{m1} \; \delta_2^{m2} \cdots \delta_p^{mp}}{m_1! \; m_2! \cdots m_p!}$$

$$\sum_{s=1}^{p} m_s = k, \qquad \sum_{s=1}^{p} s\, m_s = j$$

$$k\, T_{j,k} = \sum_{\sigma=1}^{j-k+1} \delta_\sigma \, T_{j-\sigma,\, k-1}$$

Taking (15) into account we can write the operator D as

$$D = \sum_{j=1}^{\infty} D_j, \tag{26}$$

where we put

$$D_j = X_j^* \cdot \frac{\partial}{\partial x} + Y_j^* \cdot \frac{\partial}{\partial y} + H_j^* \cdot \frac{\partial}{\partial \eta}$$

In order to abbreviate the writing we introduce the symbols

$$L_j \varphi = \sum_{k=1}^{j-1} T_{j-k} \, \varphi_k$$

$$\Lambda_j \varphi = \sum_{k=1}^{j-1} D_{j-k} \, \varphi_k$$

$$k = 1, 2, \cdots, j-1$$

$$j = 2, 3, \cdots\cdot$$

$$L_1 \varphi = 0 \qquad \Lambda_1 \varphi = 0,$$

representing the result of application of operators

$$L_j = [T_{j-1}, \; T_{j-2}, \cdots, \; T_1, \; 0, \; 0, \; 0 \cdots]$$

$$\Lambda_j = [D_{j-1}, D_{j-2}, \cdots, D_1, 0, 0, 0 \cdots]$$

to the decomposition of φ into series with respect to the small parameter.

From (4) - (6), (15), (22) - (24), (25) and (26) we deduce the set of the partial differential equations

$$\lambda \cdot \frac{\partial \mathbf{a}_j}{\partial \mathbf{y}} = \mathbf{X}_j - \mathbf{X}_j^* + \mathbf{L}_j \mathbf{X} - \Lambda_j \mathbf{a}, \tag{27}$$

$$\lambda \cdot \frac{\partial \mathbf{b}_j}{\partial \mathbf{y}} = \mathbf{Y}_j - \mathbf{Y}_j^* + \mathbf{T}_j \lambda + \mathbf{L}_j \mathbf{Y} - \Lambda_j \mathbf{b}, \tag{28}$$

$$\lambda \cdot \frac{\partial \boldsymbol{\beta}_j}{\partial \mathbf{y}} = H_j - H_j^* + \mathbf{L}_j H - \Lambda_j \boldsymbol{\beta}. \tag{29}$$

For the effects of the first order we have

$$\mathbf{X}_1^* = \mathbf{M}\mathbf{X}_1, \quad \mathbf{Y}_1^* = \mathbf{T}_1 \lambda + \mathbf{M}\mathbf{Y}_1, \quad H_1^* = \mathbf{M}H_1$$

$$\lambda \cdot \frac{\partial \mathbf{a}_1}{\partial \mathbf{y}} = P\mathbf{X}_1, \quad \lambda \cdot \frac{\partial \mathbf{b}}{\partial \mathbf{y}} = P\mathbf{Y}_1, \quad \lambda \cdot \frac{\partial \boldsymbol{\beta}_1}{\partial \mathbf{y}} = PH_1$$

and

$$\mathbf{a}_1 = QP\,\mathbf{X}_1, \quad \mathbf{b}_1 = QP\,\mathbf{Y}_1, \quad \boldsymbol{\beta}_1 = QP\,H_1$$

Taking into account that the D_j operators contain only the long period terms and that $\mathbf{a}_j$, $\mathbf{b}_j$, $\boldsymbol{\beta}_j$ contain only the short period terms, we conclude that

$$\Lambda_j \mathbf{a}, \quad \Lambda_j \mathbf{b}, \quad \Lambda_j \boldsymbol{\beta},$$

contain no long period terms and thus, in order to avoid the secular terms in $\mathbf{a}$, $\mathbf{b}$, $\boldsymbol{\beta}$, we have to put

$$\mathbf{X}_j^* = \mathbf{M}\mathbf{L}_j \mathbf{X}, \tag{30}$$

$$\mathbf{Y}_j^* = \mathbf{T}_j \lambda + \mathbf{M}\mathbf{L}_j \mathbf{Y}, \tag{31}$$

$$H_j^* = \mathbf{M}\mathbf{L}_j H \tag{32}$$

It follows from (27) - (29) and (30) - (32):

$$\mathbf{a}_j = QP(\mathbf{X}_j + L_j \mathbf{X} - \Lambda_j \mathbf{a}), \tag{33}$$

$$\mathbf{b}_j = QP(\mathbf{Y}_j + L_j \mathbf{Y} - \Lambda_j \mathbf{b}), \tag{34}$$

$$\beta_j = QP(H_j + L_j H - \Lambda_j \beta) \tag{35}$$

Evidently the equations (30) - (35) answer the question as to how the long period terms will be formed in higher approximations in the Krylov-Bogoliubov method either directly or as a result of the "cross-action" of the short period effects.

These equations can be written in a somewhat simpler form if the series for $\mathbf{X}$, $\mathbf{Y}$ and H are reduced to one term only. Then we have

$$L_j \varphi = T_{j-1} \varphi$$

and the basic equations become

$$\mathbf{X}_j^* = M T_{j-1} \mathbf{X},$$

$$\mathbf{Y}_j^* = T_j \lambda + M T_{j-1} \mathbf{Y},$$

$$H_j^* = M T_{j-1} H,$$

and

$$\mathbf{a}_j = QP(T_{j-1} \mathbf{X} - \Lambda_j \mathbf{a}),$$

$$\mathbf{b}_j = QP(T_{j-1} \mathbf{Y} - \Lambda_j \mathbf{b}),$$

$$\beta_j = QP(T_{j-1} H - \Lambda_j \beta).$$

HIGHER ORDER PERTURBATIONS IN POINCARÉ AND VON ZEIPEL METHOD

The introduction of the partial differential operators T_j and L_j permits one also to write the equations of the Poincaré and von Zeipel method in a very concise form. Let us consider the system of canonical equations:

$$\frac{d\mathbf{x}}{dt} = +\frac{\partial F}{\partial \mathbf{y}}, \quad \frac{d\mathbf{y}}{dt} = -\frac{\partial F}{\partial \mathbf{x}},$$

$$\frac{d\xi}{dt} = +\frac{\partial F}{\partial \eta}, \quad \frac{d\eta}{dt} = -\frac{\partial F}{\partial \xi}.$$

We assume that the Hamiltonian F is developable in powers of a small parameter and has the form

$$F = F_0(x) + F_1(x, \xi; y, \eta) + F_2(x, \xi; y, \eta) + \cdots\cdots \tag{36}$$

The functions F_j ($j = 1, 2, 3, \ldots$) are periodic in y and η with the period 2π in each component,

$$F_j = \sum_{n,\nu} F_{j,n,\nu}(x, \xi) \exp i\,(n \cdot y + \nu \cdot \eta).$$

We shall determine a canonical transformation

$$x = x^* + \frac{\partial S}{\partial y}, \qquad y^* = y + \frac{\partial S}{\partial x^*},$$

$$\xi = \xi^* + \frac{\partial S}{\partial \eta}, \qquad \eta^* = \eta + \frac{\partial S}{\partial \xi^*},$$

$$S(x^*, \xi^*; y, \eta) = \sum_{j=1}^{\infty} S_j(x^*, \xi^*; y, \eta),$$

in such a way that the new Hamiltonian

$$F^* = F_0^* + F_1^* + F_2^* + \cdots\cdots \tag{37}$$

does not contain the short period argument y^*.

In other words, that condition

$$F(x, \xi; y, \eta) = F^*(x^*, \xi^*; -, \eta^*) \tag{38}$$

is satisfied.

Putting:

$$h(x^*, \xi^*; y, \eta) = \frac{\partial S}{\partial y},$$

$$k(x^*, \xi^*; y, \eta) = \frac{\partial S}{\partial x^*},$$

$$\chi(x^*, \xi^*; y, \eta) = \frac{\partial S}{\partial \eta},$$

$$\kappa(\mathbf{x}^*, \xi^*; \mathbf{y}, \eta) = \frac{\partial S}{\partial \xi^*},$$

we write (38) as

$$F(\mathbf{x}^* + \mathbf{h}(\mathbf{x}^*, \xi^*; \mathbf{y}, \eta), \xi^* + \chi(\mathbf{x}^*, \xi^*; \mathbf{y}, \eta); \mathbf{y}, \eta)$$

$$= F^*(\mathbf{x}^*, \xi^*; -, \eta + \kappa(\mathbf{x}^*, \xi^*; \mathbf{y}, \eta)),$$

or, changing the notation, as

$$F(\mathbf{x} + \mathbf{h}(\mathbf{x}, \xi, \mathbf{y}, \eta), \xi + \chi(\mathbf{x}, \xi; \mathbf{y}, \eta); \mathbf{y}, \eta) \tag{39}$$

$$= F^*(\mathbf{x}, \xi; -, \eta + \kappa(\mathbf{x}, \xi; \mathbf{y}, \eta))$$

Introducing the displacement operators

$$1 + T(\mathbf{x}, \xi; \mathbf{y}, \eta) = \exp\left\{\mathbf{h}(\mathbf{x}, \xi; \mathbf{y}, \eta)\cdot\frac{\partial}{\partial \mathbf{x}} + \chi(\mathbf{x}, \xi; \mathbf{y}, \eta)\cdot\frac{\partial}{\partial \xi}\right\},$$

and

$$1 + T^*(\mathbf{x}, \xi; \mathbf{y}, \eta) = \exp\ \kappa(\mathbf{x}, \xi; \mathbf{y}, \eta)\cdot\frac{\partial}{\partial \eta},$$

we can write (39) in the form

$$\{1 + T(\mathbf{x}, \xi; \mathbf{y}, \eta)\} F(\mathbf{x}, \xi; \mathbf{y}, \eta) \tag{40}$$

$$= \{1 + T^*(\mathbf{x}, \xi; \mathbf{y}, \eta)\} F^*(\mathbf{x}, \xi; -, \eta)$$

Let us define the operators δ_k and δ_k^* by means of the equations

$$\delta_k = \frac{\partial S_k}{\partial \mathbf{y}}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{\partial S_k}{\partial \eta}\cdot\frac{\partial}{\partial \xi}$$

$$\delta_k^* = \frac{\partial S_k}{\partial \xi}\cdot\frac{\partial}{\partial \eta}.$$

Then we have, similarly as before,

$$1 + T = \exp \sum_{j=1}^{\infty} \delta_j = \sum_{j=0}^{\infty} T_j, \tag{41}$$

$$1 + T^* = \exp \sum_{j=1}^{\infty} \delta_j^* = \sum_{j=0}^{\infty} T_j^*, \tag{42}$$

$$L_j = [T_{j-1}, T_{j-2}, \cdots, T_1, 0, 0, \cdots]$$

$$L_j^* = [T_{j-1}^*, T_{j-2}^*, \cdots, T_1^*, 0, 0, \cdots].$$

The operators T_j are expressible in terms of δ_k, and the operators T_j^* in terms of δ_k^*, by means of the formulae given in the previous section. Making use of (37), (38), (41), and (42) we obtain:

$$\sum_{j=0}^{\infty} \sum_{k=0}^{j} T_{j-k} F_k = \sum_{j=0}^{\infty} \sum_{k=0}^{j} T_{j-k}^* F_k^*,$$

or

$$F_0^* = F_0(x),$$

$$T_j^* F_0^* = 0, \quad j > 0,$$

and

$$\sum_{k=0}^{j} T_{j-k} F_k = \sum_{k=1}^{j} T_{j-k}^* F_k^* \tag{43}$$

$$j = 1, 2, 3, \cdots$$
$$k = 0, 1, \cdots, j$$

Taking into account

$$T_j F_0 = \frac{\partial S_j}{\partial y} \cdot \frac{\partial F_0}{\partial x} + (T_j - \delta_j) F_0,$$

we can write the equation (43) as

$$\lambda \cdot \frac{\partial S_j}{\partial y} + \Phi_j = F_j^*, \tag{44}$$

where

$$\lambda = \frac{\partial F_0}{\partial x},$$

and

$$\Phi_j = F_j + (T_j - \delta_j) F_0 + L_j F - L_j^* F^*.$$

From the system of the linear partial differential equations (44) we can determine the functions S_j and F_j^* step by step. In order to dispose of the secular terms in S_j we have to put

$$F_j^* = M\,\Phi_j,$$

then we obtain:

$$S_j = QP\,\Phi_j.$$

The system of the transformed equations becomes

$$\frac{dx^*}{dt} = +\frac{\partial F^*}{\partial y^*} = 0, \qquad \frac{dy^*}{dt} = -\frac{\partial F^*}{\partial x^*} \tag{45}$$

$$\frac{d\xi^*}{dt} = +\frac{\partial F^*}{\partial \eta^*}, \qquad \frac{d\eta^*}{dt} = -\frac{\partial F^*}{\partial \xi^*}. \tag{46}$$

Besides the integral of energy the new system also possesses the integral

$$x^* = \text{const}.$$

The system (46) can be integrated independently from the system (45) and after the integration the angle y^* can be obtained by a plain quadrature.

A further reduction is possible if F^* can be re-arranged in such a way that the order of the purely secular term is lower than the order of the periodic terms. By applying the process of elimination of the periodic terms again we can obtain the solution of the original problem in the form of a Fourier series with the arguments linear with respect to time.

Such a reduction is possible in the case of the artificial satellite of the Earth (Brouwer, 1959), but it is not always possible in the lunar problem or in the stellar three body problem. If the close companion (the lunar orbiter) is moving in a highly eccentric orbit and the osculating plane has a high inclination toward the orbital plane of the distant comparison (Brown, 1936), then the solution in form of trigonometric series generally speaking cannot be obtained.

CONCLUSION

The method of Krylov and Bogoliubov does not presuppose that the forces must be conservative. Thus, the importance and the generality of this method are quite evident. The system of the differential operators and the algorithm given here permit the computation of the higher order effects up to any order. The process is formal and from the standpoint of pure mathematics might suffer, like all astronomical theories do, from the presence of small divisors.

Recently, the method of Krylov and Bogoliubov was successfully applied by Struble (1961) and by Kyner (1965) to the problem of motion of the artificial satellite. The author of the present paper has applied it to the problem of the motion of a lunar orbiter. The exposition of his results will appear in a later article.

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